# A brief survey of non-abelian tensor products of groups

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Abstract: We survey work on non-abelian tensor products of groups, with an emphasis on non-abelian tensor squares, including both general structure results and methods for computing such groups.

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# Notation and compatible action

## Notation ("driving on the left")

Notation is multiplicative

Conjugation on the left: ${}^{b}a = bab^{-1}$

Action of one group on another is also written on the left

Commutator: $[a, b] = aba^{-1}b^{-1}$

## Definition

Two groups $G$ and $H$ *act compatibly* on each other if

$$({}^{g}h)g' = {}^{g}({}^{h}({}^{g^{-1}}g')) \quad \text{and} \quad ({}^{h}g)h' = {}^{h}({}^{g}({}^{h^{-1}}h'))$$

for all $g, g' \in G$, $h, h' \in H$.

We always understand a group to be acting on itself by conjugation.

# Definition of a non-abelian tensor product/square

**Definition**

Let $G$ and $H$ be two groups that act compatibly on each other and on themselves by conjugation. The *non-abelian tensor product* $G \otimes H$ is the group generated by the symbols $g \otimes h$, where $g \in G, h \in H$, subject to the relations

$$gg' \otimes h = ({}^{g}g' \otimes {}^{g}h)(g \otimes h) \quad \text{and} \quad g \otimes hh' = (g \otimes h)({}^{h}g \otimes {}^{h}h')$$

for all $g, g' \in G$ and all $h, h' \in H$.

**Example**

Given two groups $G$ and $H$, acting on each other trivially, then $G \otimes H$ is the usual tensor product of the abelianizations, $G^{ab} \otimes_{\mathbb{Z}} H^{ab}$.

**Definition**

The group $G \otimes G$, where all actions are conjugation, is called the *non-abelian tensor square of* $G$.

# Origins: Topological connection

Brown and Loday (1987) starts with:

"Let $X$ be a pointed space and $\{A, B\}$ be an open cover of $X$ such that $A, B$ and $C = A \cap B$ are connected, and $(A, C), (B, C)$ are 1-connected. One of the corollaries of the main theorem of §5 is an algebraic description of the third triad homotopy group:

$$\pi_3(X; A, B) \cong \pi_2(A, C) \otimes \pi_2(B, C)$$

where $\otimes$ means "non-Abelian tensor product" of the two relative homotopy groups, each acting on the other via $\pi_1 C$."

# Origins: Notable papers

Foundations:

J.H.C. Whitehead. A certain exact sequence, *Ann. of Math. (2)* **52** (1950), 51–110.

C. Miller. The second homology group of a group; relations among commutators, *Proc. Amer. Math. Soc.* **3** (1952), 588–595.

R.K. Dennis. In search of "new" homology functors having a close relationship to $K-$theory, Preprint, Cornell University, Ithaca, NY 1976.

First definitions:

R. Brown, J.L. Loday. Excision homotopique en basse dimension., *C.R. Acad. Sci. Pars Sér. I Math* **298** (1984), no. 15, 353–356.

R. Brown, J.L. Loday. Van Kampen theorems for diagram of spaces., *Topology* **26** (1987), no. 3, 311–335. With an appendix by M. Zisman.

As group theoretic objects:

R. Brown, D.L. Johnson, E.F. Robertson. Some computations of nonabelian tensor products of groups, *J. Algebra* **111** (1987), no. 1, 177–202.

R. Aboughazi. Produit tensoriel du group d'Heisenberg, *Bull. Soc. Math. France* **115** (1987), 95–106.

G.J. Ellis. The nonabelian tensor product of finite groups is finite, *J. Algebra* **111** (1987), 203–205.

D.L. Johnson. The nonabelian tensor square of a finite split metacyclic group, *Proc. Edinburgh Math. Soc.* **30** (1987), 91–96.

# Some general identities (Brown, Loday)

The compatibility condition and the defining relations yield an action of $k \in G * H$ on $G \otimes H$, with ${}^k(g \otimes h) = {}^k g \otimes {}^k h$ for $g \in G, h \in H$.

## Proposition

*The following hold for all $g, g' \in G, h, h' \in H$:*

(a) ${}^g(g^{-1} \otimes h) = (g \otimes h)^{-1} = {}^h(g \otimes h^{-1})$,

(b) $(g \otimes h)(g' \otimes h')(g \otimes h)^{-1} = {}^{[g,h]}(g' \otimes h')$,

(c) $(g\, {}^h g^{-1}) \otimes h' = (g \otimes h)\, {}^{h'}(g \otimes h)^{-1}$,

(d) $g' \otimes ({}^g h h^{-1}) = {}^{g'}(g \otimes h)(g \otimes h)^{-1}$,

(e) $[g \otimes h, g' \otimes h'] = (g\, {}^h g^{-1}) \otimes ({}^{g'} h' h'^{-1})$.

# Crossed pairing (Brown, Loday)

### Definition

Let $G, H$ and $L$ be groups. A function $\phi : G \times H \to L$ is called a *crossed pairing* if for all $g, g' \in G$ and all $h, h' \in H$,

$$\phi(gg', h) = \phi({}^{g}g', {}^{g}h)\phi(g, h),$$

$$\phi(g, hh') = \phi(g, h)\phi({}^{h}g, {}^{h}h').$$

A crossed pairing determines a unique homomorphism of groups $\phi^* : G \otimes H \to L$ so that $\phi^*(g \otimes h) = \phi(g, h)$ for all $g \in G$ and $h \in H$.

$$\begin{array}{ccc} G \times H & \xrightarrow{\iota} & G \otimes H \\ & \searrow^{\phi} & \downarrow{\scriptstyle \phi^*} \\ & & L \end{array}$$

# General results (Brown, Loday)

## Proposition

*Suppose that $G, H$ are groups that act compatibly on each other.*

(a) *Suppose that $\theta : G \to A, \phi : H \to B$ are homomorphisms of groups, that $A, B$ also act compatibly on each other, and that $\theta, \phi$ preserve the actions, that is,*

$$\phi({}^{g}h) = {}^{\theta g}(\phi h) \quad \text{and} \quad \theta({}^{h}g) = {}^{\phi h}(\theta g)$$

*for all $g \in G, h \in H$. Then there is a unique homomorphism $\theta \otimes \phi : G \otimes H \to A \otimes B$ such that $(\theta \otimes \phi)(g \otimes h) = \theta g \otimes \phi h$ for all $g \in G, h \in H$. Further, if $\theta, \phi$ are onto, so also is $\theta \otimes \phi$.*

(b) *There is a unique isomorphism $\tau : G \otimes H \to H \otimes G$ such that $\tau(g \otimes h) = (h \otimes g)^{-1}$ for all $g \in G$.*

# General results (Brown, Loday)

### Proposition

*Suppose that $G, H$ are groups that act compatibly on each other.*

(a) *There are homomorphisms of groups $\lambda : G \otimes H \to G$, $\lambda' : G \otimes H \to H$ such that*

$$\lambda(g \otimes h) = g\,{}^{h}g^{-1}, \lambda'(g \otimes h) = {}^{g}h h^{-1}.$$

(b) *The crossed module rules hold for $\lambda$ and $\lambda'$, that is,*

$$\lambda({}^{g}t) = g(\lambda(t))g^{-1} \quad \text{and} \quad tt_1t^{-1} = {}^{\lambda(t)}t_1$$

*hold for all $t, t_1 \in G \otimes H, g \in G$ (and similarly for $\lambda'$).*

(c) $\lambda(t) \otimes h = t\,{}^{h}t^{-1}, g \otimes \lambda' t = {}^{g}t t^{-1}$, *and thus* $\lambda(t) \otimes \lambda'(t_1) = [t, t_1]$ *for all* $t, t_1 \in G \otimes H, g \in G, h \in H$. *Hence $G$ acts trivially on* $\ker \lambda'$ *and $H$ acts trivially on* $\ker \lambda$.

# Non-abelian tensor square: commutator map

Now consider the non-abelian tensor square $G \otimes G$.

The commutator map $[\ ,\ ] : G \times G \to G$ (a crossed pairing) induces a homomorphism of groups $\kappa : G \otimes G \to G$, such that $\kappa(g \otimes h) = [g, h]$.

$$\begin{array}{ccc} G \times G & \xrightarrow{\ \iota\ } & G \otimes G \\ & \searrow^{[\ ,\ ]} & \downarrow \kappa \\ & & G \end{array}$$

We write $J_2(G)$ for $\ker \kappa$.

If $G$ is perfect then $\kappa : G \otimes G \to G$ is the universal central extension of $G$.

# Non-abelian exterior square

Let $\nabla(G) = \langle g \otimes g \mid g \in G \rangle \leq G \otimes G$.

In fact, $\nabla(G) \leq Z(G \otimes G)$.

## Definition

The *non-abelian exterior square* $G \wedge G$ of the group $G$ is the factor group $G \otimes G/\nabla(G)$.The image of a simple tensor $g \otimes g'$ is written $g \wedge g'$.

Let $\kappa' : G \wedge G \longrightarrow G'$ be the map induced by $\kappa$.

The kernel of $\kappa'$ is the Schur multiplier $H_2(G)$ (Miller).

# Commutative diagram

(based on Brown and Loday (1984, 1987))

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
H_3(G) & \longrightarrow & \Gamma(G^{\mathrm{ab}}) & \xrightarrow{\psi} & J_2(G) & \longrightarrow & H_2(G) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \nabla(G) & \longrightarrow & G \otimes G & \longrightarrow & G \wedge G & \longrightarrow & 1 \\
 & & & & \kappa \downarrow & & \kappa' \downarrow & & \\
 & & & & G' & = & G' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}
$$

where all the sequences are exact and the short exact sequences are central. $\Gamma(G^{\mathrm{ab}})$ is Whitehead's universal quadratic functor.

# Whitehead's universal quadratic functor

Let $A$ be an abelian group. Define $\Gamma(A)$ to be the abelian group with generating set $\{\gamma a | a \in A\}$ and additional relations

$$\gamma(a^{-1}) = \gamma a, \text{ for } a \in A \text{ and}$$

$$\gamma(abc)\,\gamma a\,\gamma b\,\gamma c = \gamma(ab)\gamma(bc)\gamma(ca), \text{ for } a, b, c \in A.$$

## Proposition

(a) $\Gamma(A \times B) \cong \Gamma A \times \Gamma B \times (A \otimes B)$

(b) $\Gamma \mathbb{Z}_n \cong \begin{cases} \mathbb{Z}_n & \text{for } n \text{ odd} \\ \mathbb{Z}_{2n} & \text{for } n \text{ even} \end{cases}$ *(where $\mathbb{Z}_0 = \mathbb{Z}$)*

Thus $\Gamma(A)$ is easily computed for $A$ finitely generated.

$\psi : \Gamma(G^{\mathrm{ab}}) \to J_2(G) \leq G \otimes G$ is given by $\psi(\gamma g G') = g \otimes g$.

# Consequences of the commutative diagram

### Proposition (Brown and Loday; also Ellis (1987))

(a) *If $G$ is a finite group, then so is $G \otimes G$.*

(b) *If $G$ is a finite $p-$group, then so is $G \otimes G$.*

Proof of (a): For $G$ finite, both $H_2(G)$ and $\Gamma(G^{\mathrm{ab}})$ are finite, hence so is $J_2(G)$. Thus $G \otimes G$ is finite.

### Proposition (Brown and Loday)

*If $G$ is a free group, then $G \otimes G \cong G' \times \Gamma(G^{\mathrm{ab}})$.*

If $G$ is free of finite rank $n \geq 2$, then $G'$ is free of countably infinite rank and $\Gamma(G^{\mathrm{ab}})$ is free abelian of rank $\frac{n(n+1)}{2}$.

# Consequences of the commutative diagram: perfect groups

## Proposition

*Let $G$ be any group and let*

$$1 \longrightarrow A \xrightarrow{\iota} K \xrightarrow{\pi} G \longrightarrow 1,$$

*be a central extension. Then there is a homomorphism $\xi : G \otimes G \to K$ such that $\pi\xi$ is the commutator map $\kappa$. If $G$ is perfect, then $\xi$ is unique.*

## Definition

A *covering group* $\hat{G}$ of a group $G$ is a central extension

$$1 \longrightarrow H_2(G) \xrightarrow{\iota} \hat{G} \longrightarrow G \longrightarrow 1,$$

where $\mathrm{Im}\ \iota \subseteq \hat{\mathrm{G}}'$.

## Proposition ("Corollary 1" of Brown, Johnson and Robertson)

*When $G$ is a perfect group, $G \otimes G$ is the (unique) covering group $\hat{G}$ of $G$.*

# Consequences of the commutative diagram

**Proposition ("Corollary 2" of Brown, Johnson and Robertson)**

*If $\hat{G}$ is a covering group of $G$, then there is a map $\eta : G \wedge G \to \hat{G}'$, which is an isomorphism if $H_2(G)$ is finitely generated.*

**Proposition**

*If $G$ is a group in which $G'$ has a cyclic complement $C$, then $G \otimes G \cong (G \wedge G) \times C$.*

# Computing non-abelian tensor squares

## Using the definition

Brown, Johnson and Robertson's approach to computing a non-abelian tensor square for a finite group $G$: form the finite presentation given in the definition and to use software to perform Tietze transformations to simplify the presentation. Examine this simplified presentation to determine the isomorphism type of $G \otimes G$.

They applied this technique to all non-abelian groups of order up to 48, and classified the non-abelian tensor squares of several general types of groups.

# Computing non-abelian tensor squares

## Using the definition

Example: $G = A_4 = \langle a, b | a^3 = b^2 = (ab)^3 = 1 \rangle$.
$G \otimes G$ has 144 generators:

$$a \otimes a, a \otimes a^2, a \otimes b, a \otimes ab, \ldots, a^2 b \otimes ab, \ldots$$

and 3456 relations

$$a^2 \otimes b = ({}^a a \otimes {}^a b)(a \otimes b), \ldots, ab \otimes ab = (ab \otimes a)({}^a ab \otimes {}^a b), \ldots.$$

Using Tietze transformations and coset enumeration, determine

$$\begin{aligned} G \otimes G &\cong \langle x_1, x_2, x_3 | x_1^3 = [x_1, x_2] = [x_1, x_3] = 1, x_2^2 = x_3^2, x_2 x_3 x_2 = x_3 \rangle \\ &\cong \mathbb{Z}_3 \times Q_2, \end{aligned}$$

where $x_1 = a \otimes a, x_2 = a \otimes b, x_3 = a \otimes a^{-1} ba$.

This method becomes impractical for large finite groups since one starts with $|G|^2$ generators and $2|G|^3$ relations.

# Computing non-abelian tensor squares

## Using the definition

### Proposition

*Let $Q_m$ be the quartionic group of order $4m$ (with presentation $\langle x, y | y^m = x^2, xyx^{-1} = y^{-1}\rangle$). Then*

$$Q_m \otimes Q_m \cong \begin{cases} \mathbb{Z}_4 \times \mathbb{Z}_m & \text{for } m \text{ odd} \\ \mathbb{Z}_2 \times \mathbb{Z}_{2m} \times \mathbb{Z}_{2+k} \times \mathbb{Z}_2 & \text{for } m = 4r + k, k \in \{0, 2\} \end{cases}$$

### Proposition (also see Aboughazi)

*Let $D_m$ be the dihedral group of order $2m$ (with presentation $\langle x, y | y^m = 1, xyx^{-1} = y^{-1}\rangle$). Then*

$$D_m \otimes D_m \cong \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_m & \text{for } m \text{ odd} \\ \mathbb{Z}_2 \times \mathbb{Z}_m \times \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{for } m \text{ even} \end{cases}$$

# Computing non-abelian tensor squares

## Using crossed pairings

Recall: A crossed pairing $\phi : G \times H \to L$ determines a unique homomorphism of groups $\phi^* : G \otimes H \to L$ so that $\phi^*(g \otimes h) = \phi(g, h)$ for all $g \in G$ and $h \in H$.

$$\begin{array}{ccc} G \times H & \xrightarrow{\iota} & G \otimes H \\ & \searrow^{\phi} & \downarrow{\phi^*} \\ & & L \end{array}$$

Method: Conjecture a group $L$ for $G \otimes G$, as well as a map $\phi : G \times G \to L$. Show that $\phi$ is a crossed pairing and that the induced map $\phi^*$ is an isomorphism.

# Computing non-abelian tensor squares

## Crossed pairings: sample result

### Theorem (Bacon, Kappe and Morse (1997); B, Morse and Redden (2004))

(a) *The non-abelian tensor square of the free* $2-$*Engel group of rank* $2$ *is free abelian of rank* $6$.

(b) *The non-abelian tensor square of the free* $2-$*Engel group of rank* $n > 2$ *is a direct product of a free abelian group of rank* $\frac{1}{3}n(n^2+2)$ *and an* $n(n-1)-$*generated nilpotent group of class* $2$ *whose derived subgroup has exponent* $3$.

The proofs involve very detailed computer-assisted calculations, sufficient to dissuade attempting to use crossed pairings to investigate non-abelian tensor squares of (e.g.) finite rank free nilpotent groups of class 3.

### Definition (Ellis and Leonard (1995), Rocco (1991))

Let $G$ be a group with presentation $\langle \mathcal{G} | \mathcal{R} \rangle$ and let $G^{\varphi}$ be an isomorphic copy of $G$ via the mapping $\varphi : g \to g^{\varphi}$ for all $g \in G$. Define the group $\nu(G)$ to be

$$\nu(G) = \langle \mathcal{G}, \mathcal{G}^{\varphi} | \mathcal{R}, \mathcal{R}^{\varphi}, {}^{x}[g, h^{\varphi}] = [{}^{x}g, ({}^{x}h)^{\varphi}] = {}^{x^{\varphi}}[g, h^{\varphi}], \forall x, g, h \in G \rangle.$$

The groups $G$ and $G^{\varphi}$ embed isomorphically into $\nu(G)$. By convention the labels $G$ and $G^{\varphi}$ also denote their natural isomorphic copies in $\nu(G)$.

# Computing non-abelian tensor squares

## The group $\nu(G)$

### Theorem (Ellis and Leonard (1995), Rocco (1991))

*Let $G$ be a group. The map*

$$\phi : G \otimes G \to [G, G^{\varphi}] \lhd \nu(G)$$

*defined by $\phi(g \otimes h) = [g, h^{\varphi}]$ for all $g$ and $h$ in $G$ is an isomorphism.*

Note that $\nu(G)$ has $2|\mathcal{G}|$ generators, a significant reduction from the number of generators of $G \otimes G$. Ellis and Leonard show that the number of relations for $\nu(G)$ can be pruned to a degree that depends on the size and structure of the center of $G$.

Strategy: compute a small finite presentation for $\nu(G)$ and use it to determine its subgroup $[G, G^{\varphi}]$.

## Theorem (Rocco (1991))

*Let $G$ be a group.*

(a) *If $G$ is finite then $\nu(G)$ is finite.*

(b) *If $G$ is a finite p-group then $\nu(G)$ is a finite p-group.*

(c) *If $G$ is nilpotent of class $c$ then $\nu(G)$ is nilpotent of class at most $c+1$.*

(d) *If $G$ is solvable of derived length $d$ then $\nu(G)$ is solvable of derived length at most $d+1$.*

(e) *Let $\iota : [G, G^{\varphi}] \to \nu(G)$ be the natural inclusion map and let $\xi : \nu(G) \to G \times G$ be the homomorphic extension of the map sending the generator $g \in G$ of $\nu(G)$ to $(g, 1)$ and the generator $g^{\varphi} \in G^{\varphi}$ of $\nu(G)$ to $(1, g)$. Then*

$$1 \longrightarrow [G, G^{\varphi}] \xrightarrow{\ \iota\ } \nu(G) \xrightarrow{\ \xi\ } G \times G \longrightarrow 1$$

*is a short exact sequence.*

# Properties of $\nu(G)$

## Lemma (Rocco; B, Moravec, Morse)

*Let $G$ be a group. The following relations hold in $\nu(G)$:*

(a) ${}^{[g_3,g_4^{\varphi}]}[g_1,g_2^{\varphi}] = {}^{[g_3,g_4]}[g_1,g_2^{\varphi}]$ *and* ${}^{[g_3^{\varphi},g_4]}[g_1,g_2^{\varphi}] = {}^{[g_3,g_4]}[g_1,g_2^{\varphi}]$ *for all* $g_1, g_2, g_3, g_4$ *in* $G$;

(b) $[g_1^{\varphi},g_2,g_3] = [g_1,g_2,g_3^{\varphi}] = [g_1^{\varphi},g_2,g_3^{\varphi}] = [g_1,g_2^{\varphi},g_3] = [g_1^{\varphi},g_2^{\varphi},g_3] = [g_1,g_2^{\varphi},g_3^{\varphi}]$ *for all* $g_1, g_2, g_3$ *in* $G$;

(c) $[g_1,[g_2,g_3]^{\varphi}] = [g_2,g_3,g_1^{\varphi}]^{-1}$;

(d) $[g,g^{\varphi}]$ *is central in* $\nu(G)$ *for all* $g$ *in* $G$;

(e) $[g_1,g_2^{\varphi}][g_2,g_1^{\varphi}]$ *is central in* $\nu(G)$ *for all* $g_1, g_2$ *in* $G$;

(f) $[g,g^{\varphi}] = 1$ *for all* $g$ *in* $G'$.

**Theorem (Ellis and Leonard)**

*Let $G$ be a group.*

(a) *$\nu(G)$ is isomorphic to $((G \otimes G) \rtimes G) \rtimes G$.*

(b) *Let $\widetilde{G}$ (respectively, $\widetilde{G^{\varphi}}$) denote the normal closure of $G$ (respectively, $G^{\varphi}$) in $\nu(G)$. Then*

$$G \otimes G \cong \widetilde{G} \cap \widetilde{G^{\varphi}}.$$

For $G$ a finite $p$-group, use the nilpotent quotient algorithm to compute $\nu(G)$, otherwise use coset enumeration. Then compute the subgroup $\widetilde{G} \cap \widetilde{G^{\varphi}}$.

# $G \otimes G$ for group $G$ of order 4096

Ellis and Leonard computed, for example, $G \otimes G$ for $G$ the Burnside group of exponent 4 and rank 2, a group of order 4096, by applying a $p$-quotient algorithm to find a power-conjugate presentation of $\nu(G)$, from which the subgroup $[G, G^\varphi]$ can be obtained.

$G \otimes G$ has order $2^{22}$ and is the extension of $G'$ by the abelian group $(\mathbb{Z}_4)^4 \times (\mathbb{Z}_2)^6$.

(The computation took 61 seconds of CPU time using CAYLEY.)

### Definition

A group $G$ is *polycyclic* if it has a subnormal series

$$1 = G_0 \lhd G_1 \lhd \ldots \lhd G_{n-1} \lhd G_n = G$$

with cyclic factors $G_i/G_{i-1}$ for $i = 1, \ldots, n$.

### Definition

A *polycyclic generating sequence* for a polycyclic group $G$ is a sequence $\mathfrak{G} = (g_1, g_2, \ldots, g_n)$ such that $G_i = \langle G_{i-1}, g_i \rangle$ for $i = 1, \ldots, n$.

**Theorem (B and Morse (2009))**

*Let $G$ be a polycyclic group with a finite presentation $\langle \mathcal{G} \mid \mathcal{R} \rangle$ and polycyclic generating sequence $\mathfrak{G}$. Then*

(a) *The groups $G \otimes G$ and $\nu(G)$ are polycyclic.*

(b) *The group $\nu(G)$ has a finite presentation that depends only on $\mathcal{G}$, $\mathcal{R}$ and $\mathfrak{G}$.*

(c) *The non-abelian tensor square $G \otimes G$ is generated by the set*

$$\{\mathfrak{g}^{\pm 1} \otimes \mathfrak{h}^{\pm 1} \mid \text{ for all } \mathfrak{g}, \mathfrak{h} \text{ in } \mathfrak{G}\}.$$

These results support hand and computer calculations, for example, using a polycyclic quotient algorithm.

### Theorem

*Let $\mathcal{N}_{n,c}$ denote the free nilpotent group of class $c$ and rank $n > 1$, and denote the free abelian group of rank $n$ by $F_n^{\mathrm{ab}}$.*

(a) *(Bacon (1994)) For $c = 2, \mathcal{N}_{n,2} \otimes \mathcal{N}_{n,2} \cong F_{f(n)}^{\mathrm{ab}}$, where*

$$f(n) = \frac{n(n^2 + 2n - 1)}{3}.$$

(b) *(B and Morse (2008)) For $c = 3, \mathcal{N}_{n,3} \otimes \mathcal{N}_{n,3}$ is the direct product of $W_n$ and $F_{h(n)}^{\mathrm{ab}}$, where $W_n$ is nilpotent of class 2, minimally generated by $n(n-1)$ elements, and*

$$h(n) = \frac{n(3n^3 + 14n^2 - 3n + 10)}{24}.$$

# Structure of non-abelian tensor square

## Commutative diagram revisited

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
H_3(G) & \longrightarrow & \Gamma(G^{\mathrm{ab}}) & \xrightarrow{\psi} & J_2(G) & \longrightarrow & H_2(G) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \nabla(G) & \longrightarrow & G \otimes G & \longrightarrow & G \wedge G & \longrightarrow & 1 \\
 & & & & \kappa \downarrow & & \kappa' \downarrow & & \\
 & & & & G' & = & G' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}
$$

It turns out that the middle row splits under fairly general conditions.

# Structure of non-abelian tensor square



The following technical lemma, an improvement of Proposition 3.3 of Rocco (1994), shows that the structure of $\nabla(G)$ depends on $G^{\mathrm{ab}}$.

## Lemma (B, Fumagalli and Morigi)

*Let $G$ be a group such that $G^{\mathrm{ab}}$ is finitely generated. Assume that $G^{\mathrm{ab}}$ is the direct product of the cyclic groups $\langle x_i G' \rangle$, for $i = 1, \ldots, s$ and set $E(G)$ to be $\left\langle [x_i, x_j^{\varphi}] | i < j \right\rangle [G', G^{\varphi}]$. Then the following hold.*

(a) $\nabla(G)$ *is generated by the elements of the set* $\left\{ [x_i, x_i^{\varphi}], [x_i, x_j^{\varphi}][x_j, x_i^{\varphi}] | 1 \leq i < j \leq s \right\}$.

(b) $[G, G^{\varphi}] = \nabla(G)E(G)$.

# Structure of non-abelian tensor square

## Splitting Theorem

### Theorem (B, Fumagalli and Morigi)

*Assume that $G^{\mathrm{ab}}$ is finitely generated. Then the following hold.*

(a) *The restriction $f|_{\nabla(G)} : \nabla(G) \longrightarrow \nabla(G^{\mathrm{ab}})$ of the projection $f : G \longrightarrow G^{\mathrm{ab}}$ onto $G^{\mathrm{ab}}$ has kernel $N = E(G) \cap \nabla(G)$. Moreover, $N$ is a central elementary abelian 2-subgroup of $[G, G^{\varphi}]$ of rank at most the 2-rank of $G^{\mathrm{ab}}$.*

(b) $[G, G^{\varphi}]/N \simeq \nabla(G^{\mathrm{ab}}) \times (G \wedge G)$.

(c) *Suppose either that $G^{\mathrm{ab}}$ has no elements of order two or that $G'$ has a complement in $G$. Then $\nabla(G) \cong \nabla(G^{\mathrm{ab}})$ and $G \otimes G \cong \nabla(G) \times (G \wedge G)$.*

### Corollary

*Let $G$ be a group such that $G^{\mathrm{ab}}$ is a finitely generated group with no elements of order 2. Then $J(G) \cong \Gamma(G^{\mathrm{ab}}) \times H_2(G)$.*

### Corollary (B, Moravec and Morse (2008))

*Let $G = \mathcal{N}_{n,c}$ be the free nilpotent group of class $c$ and rank $n > 1$. Then $J(G) \cong \Gamma(G^{\mathrm{ab}}) \times H_2(G)$ is free abelian of rank $\binom{n+1}{2} + M(n, c+1)$, where $M(n, c)$ denotes the number of basic commutators in $n$ symbols of weight $c$.*

# Structure of non-abelian tensor square

Recall that if the Schur multiplier $H_2(G)$ of $G$ is finitely generated, then $G \wedge G$ is isomorphic to the derived subgroup of any covering group $\hat{G}$ of $G$.

If $G$ is the free nilpotent group $\mathcal{N}_{n,c}$, then $\hat{G} \cong \mathcal{N}_{n,c+1}$, so that $G \wedge G \cong \mathcal{N}'_{n,c+1}$. We recover:

**Theorem (B, Moravec and Morse (2008))**

*Let $G = \mathcal{N}_{n,c}$ be the free nilpotent group of class $c$ and rank $n > 1$. Then*

$$G \otimes G \cong \nabla(G) \times (G \wedge G) \cong F^{\mathrm{ab}}_{\binom{n+1}{2}} \times \mathcal{N}'_{n,c+1}.$$

The structure of the derived subgroup $\mathcal{N}'_{n,c+1}$ is further examined in B, Moravec and Morse (2008a), resulting in a more refined description of $\mathcal{N}_{n,c} \otimes \mathcal{N}_{n,c}$.

# Structure of non-abelian tensor square

## Exterior square theorem

The proof of the main result of Miller (1952) can be generalized to show:

### Theorem (B, Fumagalli and Morigi)

*Let $G$ be a group and let $F$ be a free group such that $G \cong F/R$ for some normal subgroup $R$ of $F$. Then $G \wedge G \cong F'/[F, R]$.*

The earlier results of Brown, Johnson and Robertson (1987) on the non-abelian tensor squares of free groups of finite rank also follow directly from the splitting and exterior square theorems. We also obtain a result for free soluble groups.

## Corollary (B, Fumagalli and Morigi)

*Let $F$ be the free group of finite rank $n > 1$, let $d$ be a natural number, and let $G = F/F^{(d)}$ be the free solvable group $\mathcal{S}_{n,d}$ of derived length $d$ and rank $n > 1$. Then*

$$G \otimes G \cong \mathbb{Z}^{n(n+1)/2} \times F'/[F, F^{(d)}]$$

*is an extension of a nilpotent group of class $\leq 3$ by a free solvable group of derived length $d - 2$ and infinite rank.*
*In particular, if $d = 2$, then $G \otimes G$ is a nilpotent group.*

# Further results

### Theorem (Ellis and McDermott (1998))

*Let $G$ be a finite group of order $p^n$ (for $p$ prime) and let $d$ be the minimum number of generators of $G$. Then $p^{d^2} \leq |G \otimes G| \leq p^{nd}$.*

Jarafi (2016) improves the upper bound to $p^{(n-1)d+2}$.

### Theorem (Bastos and Rocco (2016))

*Let $G$ be a finite-by-nilpotent group. Then $G \otimes G$ is finite-by-nilpotent and $\nu(G)$ is nilpotent-by-finite.*

### Theorem (Bastos, Nakaoka and Rocco (2018))

*Let $G, H$ be groups acting compatibly on each other such that the set of simple tensors $g \otimes h$ is finite. Then $G \otimes H$ is finite.*

# Further results

Bastos, Rocco (2017, 2 papers) discuss non-abelian tensor squares/products for residually finite groups satisfying certain identities.

Bardakov, Lavrenov, Neshchadim (2019) give an example of a linear group with non-abelian tensor square not linear, and conditions for the linearity of non-abelian tensor products. Application to some one relator groups and some knot groups.

Ellis(1991), Niroomand (2012): study non-abelian tensor products in Lie algebras.

Thank you

# The End!