Math 244 Calculus III

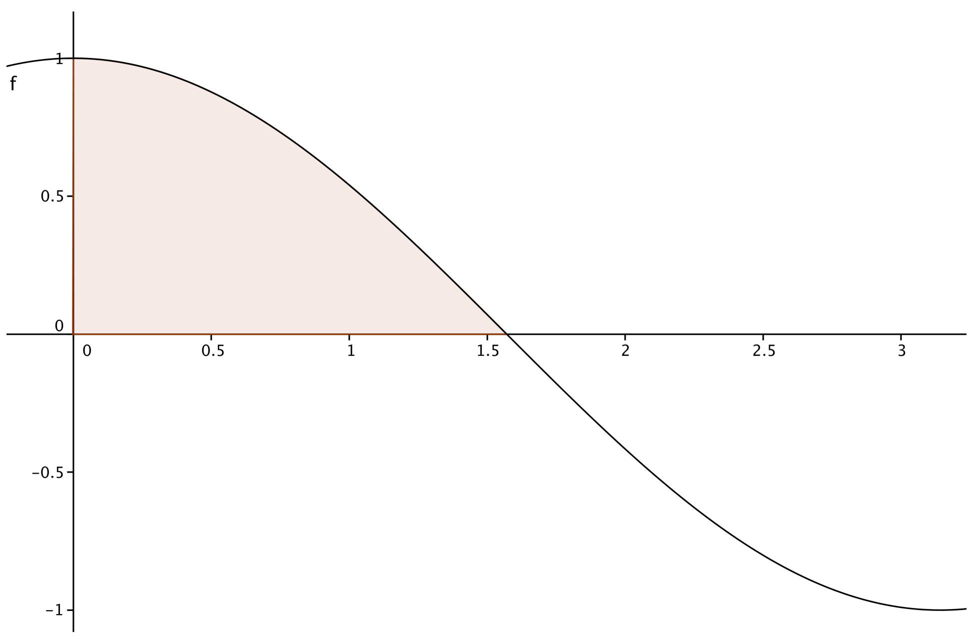
Multiple Integration Set-Up Practice — Solutions

Work in groups at the board to set up, but do not evaluate, the integrals indicated. Pay attention to the requested form of each integral.
Sketch each region of integration as part of the solution process.

1. Set up $\iint_R f(x,y)\,dA$ when R is the region bounded by the positive x-axis, the positive y-axis and $y = \cos(x)$ for $0 \le x \le \pi$. Write down the integrals for both forms $dxdy$ and $dydx$.

   Solution: The region described is bound by the two axes and $y = \cos(x)$ for $0 \le x \le \frac{\pi}{2}$. Hence the two forms of the integral are

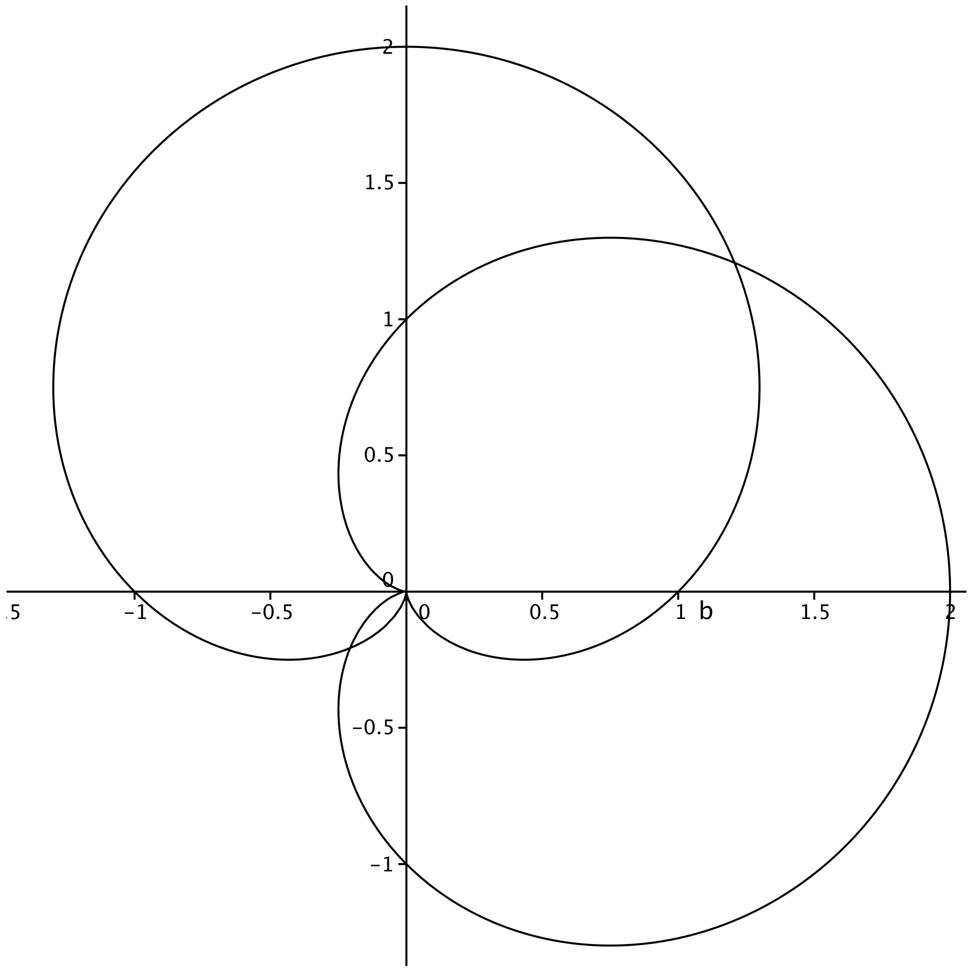
$$\int_0^{\frac{\pi}{2}} \int_0^{\cos(x)} f(x,y)\ dydx \quad \text{and} \quad \int_0^1 \int_0^{\cos^{-1}(y)} f(x,y)\ dxdy.$$



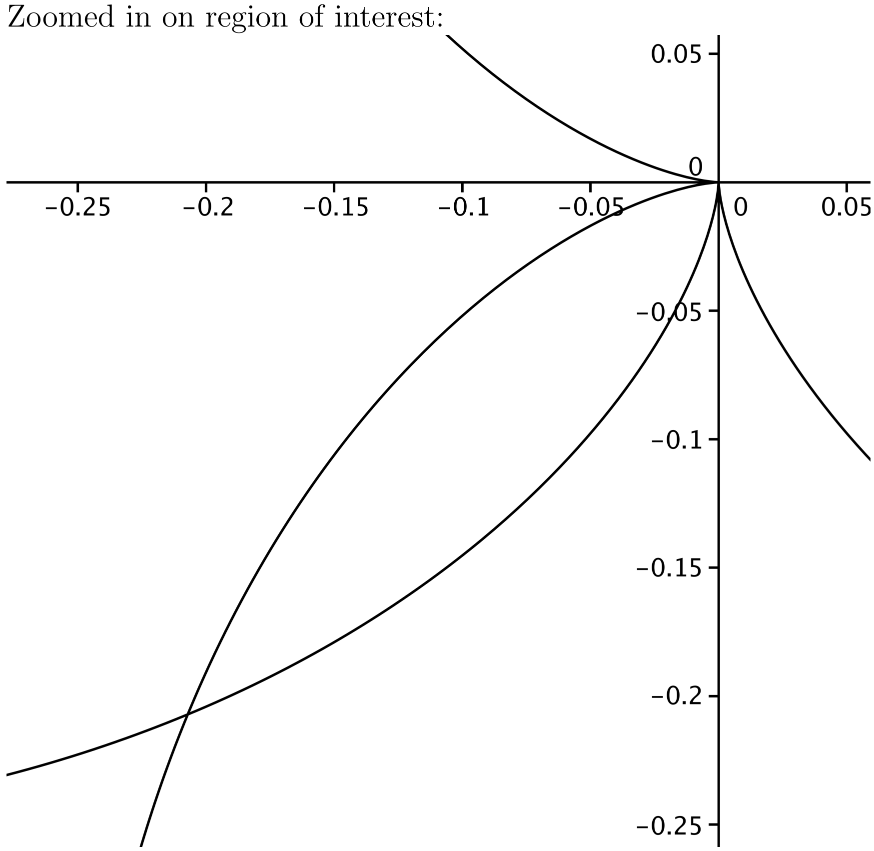
2. (Polar coordinates) Set up $\iint_R f(r,\theta)\,dA$ when R is the region bounded by the curves $r = 1+\cos(\theta)$ and $r = 1+\sin(\theta)$ in the third quadrant in the form $d\theta dr$.

   Solution: In $d\theta dr$ form the integral is

$$\int_0^{1-\frac{1}{\sqrt{2}}} \int_{2\pi-\cos^{-1}(r-1)}^{\pi-\sin^{-1}(r-1)} f(r,\theta) r\ d\theta dr$$



Zoomed in on region of interest:



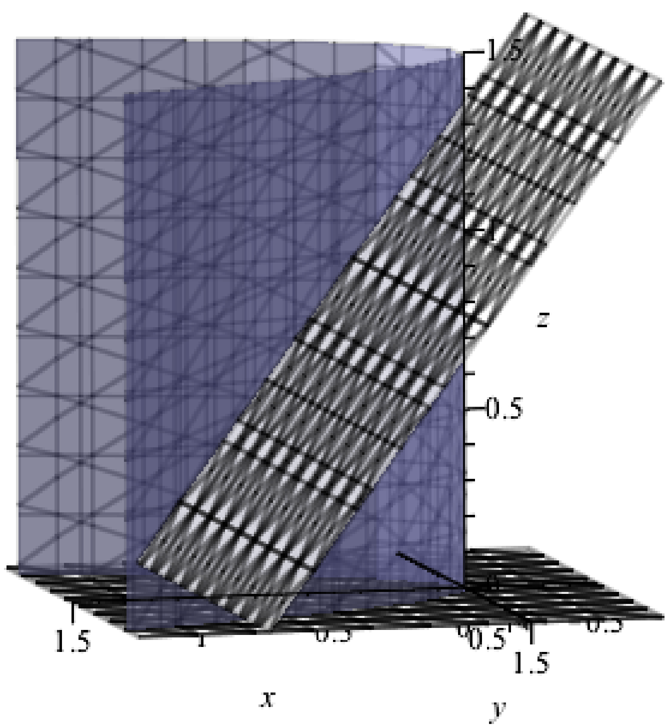
3. Set up $\iiint_W f(x,y,z)\,dV$ when W is the region bounded by the parabolic cylinder $x = y^2$ and the planes $z = 0$ and $x + z = 1$. Write down three versions of the integral: one for each of the forms $dzdxdy, dydzdx$ and $dxdzdy$. (You may find it useful to consider, respectively, the projections of the region onto the xy-plane, the xz-plane, and the yz-plane, to determine the limits for the outer two integrals in each case.)

Solution: The volume is bounded by the two planes $z = 0$ and $x + z = 1$ (x slope is -1) and the parabolic cylinder $x = y^2$. The three forms of the integral are:

$$\int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{1-x} f(x,y,z)\ dzdxdy,\ \int_{0}^{1}\int_{0}^{1-x}\int_{-\sqrt{x}}^{\sqrt{x}} f(x,y,z)\ dydzdx$$

$$\text{and}\quad \int_{-1}^{1}\int_{0}^{1-y^2}\int_{y^2}^{1-z} f(x,y,z)\ dxdzdy.$$

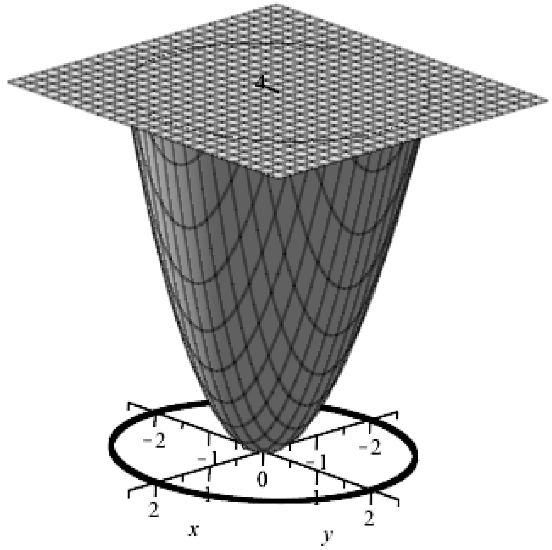
(In the third integral, for a given value of y, z runs from 0 to the maximum value of $1 - x$, which occurs when $x = y^2$, so z runs to $1 - y^2$.)



4. (Cylindrical coordinates) Set up $\iiint_W f(r,\theta,z)\,dV$ when W is the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$. Use the form $dzdrd\theta$.

Solution: The paraboloid has circular symmetry, with shadow in the xy-plane a circle of radius 2 for that part of the paraboloid lying under $z = 4$. Thus the integral has form:

$$\int_{0}^{2\pi}\int_{0}^{2}\int_{r^2}^{4} f(r,\theta,z)\ rdzdrd\theta.$$



5. Convert the following triple integral to spherical coordinates:

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2\sqrt{x^2+y^2+z^2}\ dzdxdy.$$

Solution: As y runs from -2 to 2, x runs from 0 to the "front" half of the circle $x^2+y^2=4$ in the xy-plane. For a given x, y combination, z runs from the bottom to the top of the sphere $x^2+y^2+z^2=4$. So the region is the "front" half of a sphere of radius 2. The integrand in spherical coordinates is: $(\rho\sin(\varphi)\cos(\theta))^2\sqrt{\rho^2}$ times $\rho^2\sin(\varphi)$, that is, $\rho^5\sin^3(\varphi)\cos^2(\theta)$. The integral is:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{\pi}\int_{0}^{2}\rho^5\sin^3(\varphi)\cos^2(\theta)\ d\rho d\varphi d\theta$$