Math 401 Probability Homework #4 Solutions

3.148 $m(t) = \dfrac{pe^t}{1-qe^t}$ $(q = 1-p) \Rightarrow m'(t) = \dfrac{pe^t(1-qe^t) - pe^t(-qe^t)}{(1-qe^t)^2}$

$\Rightarrow m'(t) = \dfrac{pe^t - pqe^{2t} + pqe^{2t}}{(1-qe^t)^2} = \dfrac{pe^t}{(1-qe^t)^2}$

$\Rightarrow m''(t) = \dfrac{pe^t(1-qe^t)^{\cancel{2}} - pe^t\cancel{(1-qe^t)}(-qe^t)}{(1-qe^t)^{\cancel{4}\,3}} = \dfrac{pe^t - pqe^{2t} + 2pqe^{2t}}{(1-qe^t)^3}$

$= \dfrac{pe^t + pqe^t}{(1-qe^t)^3} = \dfrac{pe^t(1+qe^t)}{(1-qe^t)^3}$

$\therefore E(Y) = m'(0) = \dfrac{p}{(1-q)^2} = \dfrac{p}{p^2} = \dfrac{1}{p}$, $E(Y^2) = m''(0) = \dfrac{p(1+q)}{(1-q)^3} = \dfrac{p(1+q)}{p^3}$

$= \dfrac{1+q}{p^2}$

$\therefore V(Y) = E(Y^2) - E(Y)^2 = \dfrac{1+q}{p^2} - \dfrac{1}{p^2} = \dfrac{q}{p^2} = \dfrac{1-p}{p^2}$.

3.158 Y has mgf $m(t)$. $W = aY+b$. Moment generating function for W is $m_W(t) = E(e^{tW}) = E(e^{t(aY+b)})$

$= \sum_y e^{t(ay+b)} p(y) = \sum_y e^{tb} e^{tay} p(y) = e^{tb} \sum e^{(at)y} p(y)$

$= e^{tb} m(at)$.

3.159 $m_W(t) = e^{tb} m(at) \Rightarrow m_W'(t) = be^{tb} m(at) + e^{tb} m'(at) a$

$\Rightarrow E(W) = m_W'(0) = be^0 m(0) + e^0 m'(0) a = b + aE(Y)$

$m_W''(t) = b^2 e^{tb} m(at) + 2be^{tb} a m'(at) + a^2 e^{tb} m''(at)$

$\Rightarrow E(W^2) = m_W''(0) = b^2 e^0 m(0) + 2be^0 a m'(0) + a^2 e^0 m''(0)$

$= b^2 + 2abE(Y) + a^2E(Y^2)$.

$\therefore V(W) = E(W^2) - E(W)^2 = (b^2 + 2abE(Y) + a^2E(Y^2)) - (b + aE(Y))^2$

$= \cancel{b^2} + \cancel{2abE(Y)} + a^2E(Y^2) - \cancel{b^2} - \cancel{2abE(Y)} - a^2E(Y)^2$

$= a^2(E(Y^2) - E(Y)^2) = a^2V(Y)$.

3.164 Prob. f^n for binomial r.v.: $p(y) = \binom{n}{y} p^y (1-p)^{n-y}$

pgf: $p(t) = E(t^Y) = \sum_{y=0}^{n} t^y \binom{n}{y} p^y (1-p)^{n-y}$

$= \sum_{y=0}^{n} \binom{n}{y} (tp)^y (1-p)^{n-y} = (tp + (1-p))^n$ (Bin. Thm.)

$\therefore p'(t) = n(tp + (1-p))^{n-1} \cdot p$

$\Rightarrow E(Y) = p'(1) = n(p + (1-p))^{n-1} p = np$

3.168 (a) Binomial experiment $p = 0.2$, $n = 100 \Rightarrow E(Y) = np = 20$.

(b) $V(Y) = npq = 100(0.2)(0.8) = 16 \Rightarrow \sigma = 4$.

(c) $\mu \pm 2\sigma = [12, 28]$, $\mu \pm 3\sigma = [8, 32]$.

(d) 50 is $\frac{50-20}{4} = 7.5$ standard deviation above the mean. It is highly unlikely that you would obtain a score at or above 50.

4.12 $f(y) = \begin{cases} 0.2 & -1 \le y \le 0 \\ 0.2 + cy & 0 < y \le 1 \\ 0 & o/w \end{cases}$

(a) $\int_{-\infty}^{\infty} f(y)\,dy = 1 \Rightarrow \int_{-1}^{0} 0.2\,dy + \int_0^1 0.2 + cy\,dy = 0.2y\Big|_{-1}^{0} + \left(0.2y + \frac{c}{2}y^2\right)\Big|_0^1$
$= (0.2 + 0.2 + \frac{c}{2} - 0) = 1 \Rightarrow \frac{c}{2} = 0.6 \Rightarrow c = 1.2$.

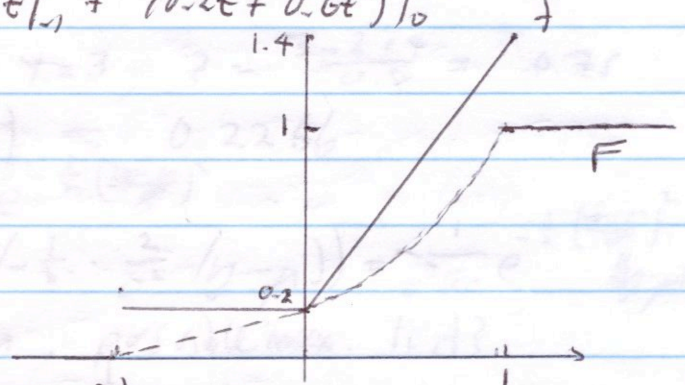
(b) $F(y) = \begin{cases} 0 & y \le -1 \\ \int_{-1}^{y} 0.2\,dt & -1 < y \le 0 \\ \int_{-1}^{0} 0.2\,dt + \int_0^y 0.2 + 1.2t\,dt & 0 < y \le 1 \\ 1 & y > 1 \end{cases}$ (*) note carefully!

$\int_{-1}^{y} 0.2\,dt = 0.2t\Big|_{-1}^{y} = 0.2y + 0.2$

$\int_{-1}^{0} 0.2\,dt + \int_0^y 0.2 + 1.2t\,dt = 0.2t\Big|_{-1}^{0} + (0.2t + 0.6t^2)\Big|_0^y$
$= 0.2 + 0.2y + 0.6y^2$

$\therefore F(y) = \begin{cases} 0 & y \le -1 \\ 0.2y + 0.2 & -1 < y \le 0 \\ 0.6y^2 + 0.2y + 0.2 & 0 < y \le 1 \\ 1 & y > 1 \end{cases}$

(c) $\longrightarrow$



(d) $F(-1) = 0$, $F(0) = 0.2$, $F(1) = 1$

(e) $P(0 \le Y \le 0.5) = F(0.5) - F(0) = (0.6(\frac{1}{2})^2 + 0.2(\frac{1}{2}) + 0.2) - (0.2) = \underline{0.25}$

(f) $P(Y > 0.1) = 1 - F(0.1) = 1 - (0.6(0.1)^2 + 0.2(0.1) + 0.2) = 0.774$
$P(Y > 0.5) = 1 - F(0.5) = 1 - 0.45 = 0.55$
$\therefore P(Y > 0.5 \mid Y > 0.1) = \frac{P((Y > 0.5) \cap (Y > 0.1))}{P(Y > 0.1)} = \frac{P(Y > 0.5)}{P(Y > 0.1)} = \frac{0.55}{0.774}$
$\approx \underline{0.7106}$

4.26 Y cont. r.v. with $E(Y) = \mu$, $V(Y) = \sigma^2$

(a) $E(aY + b) = E(aY) + E(b) = aE(Y) + b = a\mu + b$

(b) $E((aY+b)^2) = E(a^2Y^2 + 2abY + b^2) = a^2E(Y^2) + 2abE(Y) + b^2$
$\therefore V(aY+b) = E((aY+b)^2) - E(aY+b)^2 = a^2E(Y^2) + 2abE(Y) + b^2 -$
$(aE(Y) + b)^2 = a^2E(Y^2) + 2abE(Y) + b^2 - a^2E(Y)^2 - 2abE(Y) - b^2$

$= a^2\left(E(Y^2) - E(Y)^2\right) = a^2 V(Y)$.

4.32 $f(y) = \begin{cases} \frac{3}{64} y^2(4-y) & 0 \le y \le 4 \\ 0 & o/w \end{cases}$

(a) $E(Y) = \int_0^4 y f(y)\,dy = \int_0^4 \frac{3}{16}y^3 - \frac{3}{64}y^4\,dy = \frac{3}{64}y^4 - \frac{3}{320}y^5 \Big|_0^4$
$= \frac{3}{64}(256) - \frac{3}{320}(1024) = 12 - 9.6 = \underline{2.4 \text{ hrs}}$

$E(Y^2) = \int_0^4 y^2 f(y)\,dy = \int_0^4 \frac{3}{16}y^4 - \frac{3}{64}y^5\,dy = \frac{3}{80}y^5 - \frac{1}{128}y^6\Big|_0^4$
$= \frac{3}{80}(1024) - 32 = 38.4 - 32 = 6.4$

$\therefore V(Y) = E(Y^2) - E(Y)^2 = 6.4 - 2.4^2 = \underline{0.64}$.

(b) $E(200Y) = 200\,E(Y) = \$480$.

$V(200Y) = 200^2 V(Y) = 25{,}600$ (units $\$^2$) $\Rightarrow \sigma = \$160$.

(c) \$600 is only $\frac{600-480}{160} = 0.75$ std. deviations above the mean, so I'd expect this to happen occasionally.

4.50 Since the system is up 2 of the 5 hrs & the distribution of calls is uniform, $P(\text{success}) = 2/5$.

4.54 $Y = N(2.4, 0.8)$. For $Y=3$, $Z = \frac{3-2.4}{0.8} = 0.75$
$\Rightarrow P(Y>3) = P(Z>0.75) = \underline{0.2266}$

4.64 Maximize $f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}$

$f'(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}\left(-\frac{1}{2}\cdot\frac{2}{\sigma^2}(y-\mu)\right) = -\frac{1}{\sigma^3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}(y-\mu)$

$f'(y) = 0$ iff $y = \mu$, possible max. Is it?

note $f(\mu) = \frac{1}{\sigma\sqrt{2\pi}} e^0 = \frac{1}{\sigma\sqrt{2\pi}}$.

now $f''(y) = -\frac{1}{\sigma^3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}(y-\mu)\left(-\frac{(y-\mu)}{\sigma^2}\right) - \frac{1}{\sigma^3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}$

$\Rightarrow f''(\mu) = \frac{1}{\sigma^5\sqrt{2\pi}}(0) - \frac{1}{\sigma^3\sqrt{2\pi}} < 0 \therefore f$ concave down at $y=\mu \Rightarrow$ local $\underline{\text{max}}$ at $y=\mu$. This max value of f is $\underline{\frac{1}{\sigma\sqrt{2\pi}}}$, at $\underline{y=\mu}$.

4.104. $f(y) = \begin{cases} \frac{1}{100} e^{-\frac{y}{100}} & y > 0 \\ 0 & o/w \end{cases}$

$P(Y \ge 200) = \int_{200}^{\infty} \frac{1}{100} e^{-\frac{y}{100}}\,dy = -e^{-\frac{y}{100}}\Big|_{200}^{\infty} = 0 + e^{-2}$
$= e^{-2} \approx 0.1353$.

Now let Y = # lasting $\geq$ 200 hrs. Binomial with $p = e^{-2} \approx 0.1353$. ④

Want $P(Y \geq 2) = \binom{3}{2}(0.1353)^2(1-0.1353)^1 + \binom{3}{3}(0.1353)^3$

$\approx \underline{0.0500}$

4.132 (a) $\alpha = 3, \beta = 3 \Rightarrow \mu = \frac{\alpha}{\alpha+\beta} = \frac{3}{6} = \frac{1}{2}$

$\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{9}{6^2(7)} = \frac{1}{28}$

(b) $\alpha = \beta = 2 \Rightarrow \mu = 2/4 = \frac{1}{2}$

$\sigma^2 = \frac{4}{4^2(5)} = \frac{1}{20}$

(c) $\alpha = \beta = 1 \Rightarrow \mu = \frac{1}{2}$

$\sigma^2 = \frac{1}{2^2(3)} = \frac{1}{12}$.

(d) Since the $\alpha = 3 = \beta$ has the smallest variance, it will be the most consistent.