Math 401 Probability Test Two Solutions Fall 2006.

1. Binomial experiment with $p=0.3$.

(a) $n=5$, $y \geq 2$. $P(Y\geq 2) = 1 - P(Y<2) = 1-(P(Y=0)+P(Y=1))$

$= 1 - \left[\binom{5}{0}0.3^0 0.7^5 + \binom{5}{1}0.3^1 0.7^4\right] = 1-(0.16807+0.36015) = \underline{0.4718}$ (or use Tables).

(b) Geometric dist, to 1st success. $p=0.3$

$P(Y\leq 4) = 0.3 + (0.3)(0.7) + (0.3)(0.7)^2 + (0.3)(0.7)^3 = \underline{0.7599}$

2. Poisson distribution with $\lambda = 2.8$ $\qquad P(Y=y) = \dfrac{e^{-\lambda}\lambda^y}{y!}$

(a) $P(Y=0) = \dfrac{e^{-2.8}2.8^0}{0!} = e^{-2.8} \approx \underline{0.0608}$ (by Tables, 0.061)

(b) $P(Y\geq 4) = 1 - P(Y<4) \approx 1 - 0.692 = \underline{0.308}$ (by Tables) (by direct calculation $= 0.3081$)

(c) Conditional probability: $P(Y\geq 6 \mid Y\geq 2) = \dfrac{P(Y\geq 6)}{P(Y\geq 2)} = \dfrac{1-P(Y<6)}{1-P(Y<2)} = \dfrac{0.065}{0.769} \approx \underline{0.085}$ by Tables.

$((Y\geq 6)\cap(Y\geq 2)$ is event $Y\geq 6)$

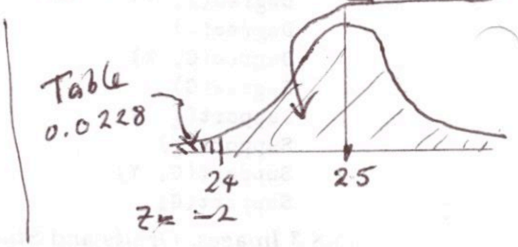
(by direct calculation, $P(Y\geq 6 \mid Y\geq 2) = \dfrac{0.06511}{0.7689} \approx 0.08468$)

3. $Y \sim N(25, 0.5)$. $\qquad z = \dfrac{y-\mu}{\sigma}$

(a) $P(Y\geq 24) = P\left(Z \geq \dfrac{24-25}{0.5}\right) = P(Z\geq -2) = 0.5 + 0.4772 = 0.9772$

(b) $P(Y<24 \mid Y<24.5) = \dfrac{P(Y<24)}{P(Y<24.5)} = \dfrac{P(Z<-2)}{P(Z<-1)}$

(the event $(Y<24)\cap(Y<24.5)$ is $Y<24$) $\qquad = \dfrac{0.0228}{0.1587} \approx \underline{0.1437}$



(c) μ unknown. Find μ so z-score for 24 gives Prob. 0.5% less than 24.

from table, $P(Z>2.575) = 0.005 \Rightarrow P(Z<-2.575) = 0.005$ also.

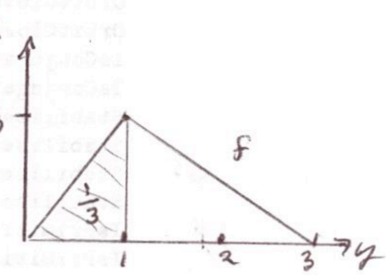
So for $Y=24$, solve $\dfrac{Y-\mu}{\sigma} = -2.575 \Rightarrow \dfrac{24-\mu}{0.5} = -2.575 \Rightarrow 24-\mu = (0.5)(-2.575)$

$\Rightarrow \mu = 24 + (0.5)(2.575) = 24 + 1.2675 \approx \underline{25.27 \text{ ounces}}$

4. (a) Solve $\int_0^1 ky\,dy + \int_1^3 \frac{k}{2}(3-y)\,dy = 1$ for k.

i.e. $\frac{k}{2}y^2\Big|_0^1 + \frac{k}{2}\left(3y - \frac{1}{2}y^2\right)\Big|_1^3 = \frac{k}{2} + \frac{k}{2}\left(\left(9-\frac{9}{2}\right)-\left(3-\frac{1}{2}\right)\right)$

$= \frac{k}{2} + \frac{k}{2}\left(9 - \frac{9}{2} - 3 + \frac{1}{2}\right) = \frac{k}{2} + \frac{k}{2}(2) = \frac{3}{2}k = 1 \Rightarrow \underline{k = \frac{2}{3}}$



(b) $F(y) = \begin{cases} 0 & y<0 \\ \int_0^y \frac{2}{3}t\,dt & 0\leq y\leq 1 \\ \frac{1}{3} + \int_1^y \frac{1}{3}(3-t)\,dt & 1<y\leq 3 \\ 1 & y>3 \end{cases}$

$\longrightarrow \int_0^y \frac{2}{3}t\,dt = \frac{1}{3}t^2\Big|_0^y = \frac{1}{3}y^2$

$\longrightarrow \frac{1}{3} + \int_1^y \frac{1}{3}(3-t)\,dt = \frac{1}{3} + \frac{1}{3}\left(3t - \frac{1}{2}t^2\right)\Big|_1^y$

$= \frac{1}{3} + \frac{1}{3}\left(\left(3y - \frac{1}{2}y^2\right) - \left(3 - \frac{1}{2}\right)\right) = \frac{1}{3} + y - \frac{1}{6}y^2 - \frac{5}{6}$

$= y - \frac{1}{6}y^2 - \frac{1}{2}$

$$\therefore F(y) = \begin{cases} 0 & y < 0 \\ \frac{1}{3}y^2 & 0 \le y \le 1 \\ y - \frac{1}{6}y^2 - \frac{1}{2} & 1 < y \le 3 \\ & y > 3 \end{cases}$$

(c) $P(1 < Y < 2) = F(2) - F(1) = \left(2 - \frac{4}{6} - \frac{1}{2}\right) - \left(\frac{1}{3}\right) = 2 - \frac{4+3+2}{6}$

$= 2 - \frac{9}{6} = \underline{\frac{1}{2}}$

5. $f(y) = \begin{cases} k e^{-y} & 1 \le y \le 2 \\ 0 & o/w \end{cases}$.

(a) Solve $\int_1^2 k e^{-y} dy = 1$ for k. $\int_1^2 k e^{-y} dy = -k e^{-y}\Big|_1^2 = -ke^{-2} + ke^{-1}$

$= k(e^{-1} - e^{-2})$

$\therefore k = \underline{\underline{\frac{1}{e^{-1} - e^{-2}}}} = \underline{\underline{\frac{e^2}{e-1}}}$ (≈ 4.3003)

(b) $m_Y(t) = E(e^{tY}) = \int_1^2 e^{ty} \frac{e^2}{e-1} e^{-y} dy = \frac{e^2}{e-1} \int_1^2 e^{y(t-1)} dy = \frac{e^2}{e-1} \frac{1}{t-1} e^{y(t-1)}\Big|_1^2$

$= \frac{e^2}{e-1} \frac{1}{t-1} \left(e^{2(t-1)} - e^{t-1}\right) = \underline{\underline{\frac{e^{2t} - e^{t+1}}{(e-1)(t-1)}}}$

(c) $E(Y) = m'(0)$. $m'(t) = \frac{1}{e-1} \frac{(2e^{2t} - e^{t+1})(t-1) - 1(e^{2t} - e^{t+1})}{(t-1)^2}$

$m'(t) = \frac{1}{e-1} \frac{2te^{2t} - te^{t+1} - 2e^{2t} + e^{t+1} - e^{2t} + e^{t+1}}{(t-1)^2} = \frac{1}{e-1} \frac{2te^{2t} - te^{t+1} - 3e^{2t} + 2e^{t+1}}{(t-1)^2}$

$E(Y) = m'(0) = \frac{1}{e-1} \frac{-3+2e}{1} = \underline{\underline{\frac{2e-3}{e-1}}}$ (≈ 1.48) (May also find $E(Y)$ by direct calc: $\int_1^2 y \frac{e^2}{e-1} e^{-y} dy$).