Math 401 Probability Test Three Solutions

1\. (a) There are probability jumps of 0.2 at $y=0$ and 0.3 at $y=1$, so the total discrete probability is 0.5 & the total continuous probability is $0.5 = 1-0.5$. Each of F_1 & F_2 is scaled up to a full prob. dist. by doubling.

Discrete: $F_1(y) = \begin{cases} 0 & y<0 \\ 0.4 & 0 \le y < 1 \\ 1 & y \ge 1 \end{cases}$ Continuous: $F_2(y) = \begin{cases} 0 & y<0 \\ y^2 & 0 \le y < 1 \\ 1 & y \ge 1 \end{cases}$ (remove the prior 0.2 first)

$F(y) = 0.5\, F_1(y) + 0.5\, F_2(y)$.

(b) Discrete: $p_1(y) = \begin{cases} 0.4 & y=0 \\ 0.6 & y=1 \end{cases}$ Cont.: $f_1(y) = \frac{d}{dy} F_1(y) = \begin{cases} 2y & 0<y<1 \\ 0 & \text{o/wise} \end{cases}$

(c) $E(Y) = c_1 E(Y_1) + c_2 E(Y_2) = 0.5 E(Y_1) + 0.5 E(Y_2)$

$E(Y_1) = (0)(0.4) + (1)(0.6) = 0.6$, $E(Y_2) = \int_0^1 y \cdot 2y\, dy = \int_0^1 2y^2 dy = \frac{2}{3} y^3 \Big|_0^1 = \frac{2}{3}$

$\therefore E(Y) = \frac{1}{2}\frac{3}{5} + \frac{1}{2}\frac{2}{3} = \frac{3}{10} + \frac{1}{3} = \frac{9+10}{30} = \underline{\frac{19}{30}}$

2 (a) $f_1(y_1) = \int_0^{y_1} \frac{3}{10} y_1 (2-y_2)\, dy_2 = \frac{3}{10} y_1 \left(2y_2 - \frac{1}{2}y_2^2\right)\Big|_0^{y_1} = \begin{cases} \frac{3}{10} y_1 \left(2y_1 - \frac{1}{2} y_1^2\right), & 0<y_1<2 \\ 0 & \text{o/wise} \end{cases}$

(b) $f_2(y_2) = \int_{y_2}^{2} \frac{3}{10} y_1 (2-y_2)\, dy_1 = \frac{3}{10}(2-y_2)\frac{1}{2} y_1^2 \Big|_{y_2}^{2} = \begin{cases} \frac{3}{20}(2-y_2)(4-y_2^2) & 0<y_2<2 \\ 0 & \text{o/wise} \end{cases}$

(c) $f(y_1 | y_2) = \dfrac{f(y_1,y_2)}{f_2(y_2)} = \dfrac{\frac{3}{10} y_1 (2-y_2)}{\frac{3}{20}(2-y_2)(4-y_2^2)} = \begin{cases} \dfrac{2y_1}{4-y_2^2} & y_2 < y_1 < 2 \\ 0 & \text{o/wise} \end{cases}$

(d) $P(0<Y_1<1) = \int_0^1 f_1(y_1)\, dy_1 = \int_0^1 \frac{3}{10}\left(2y_1^2 - \frac{1}{2} y_1^3\right) dy_1 = \frac{3}{10}\left(\frac{2}{3} y_1^3 - \frac{1}{8} y_1^4\right)\Big|_0^1$

$= \frac{3}{10}\left(\frac{2}{3} - \frac{1}{8}\right) = \frac{3}{10} \cdot \frac{16-3}{24} = \underline{\frac{13}{80}}$

(e) $P(Y_1 < \frac{3}{2} | Y_2 = 1) = \int_1^{3/2} f(y_1 | y_2 = 1)\, dy_1 = \int_1^{3/2} \frac{2y_1}{3} dy_1 = \frac{1}{3} y_1^2 \Big|_1^{3/2} = \frac{1}{3}\left(\frac{9}{4} - 1\right) = \frac{1}{3}\left(\frac{5}{4}\right) = \underline{\frac{5}{12}}$.

(f) $E(Y_1) = \int_0^2 y_1 f_1(y_1)\, dy_1 = \int_0^2 y_1 \frac{3}{10}\left(2y_1^2 - \frac{1}{2} y_1^3\right) dy_1 = \frac{3}{10}\int_0^2 2y_1^3 - \frac{1}{2} y_1^4\, dy_1$

$= \frac{3}{10}\left(\frac{1}{2} y_1^4 - \frac{1}{10} y_1^5\right)\Big|_0^2 = \frac{3}{10}\left(8 - \frac{32}{10}\right) = \frac{3}{10} \cdot \frac{40-16}{5} = \frac{3}{10} \cdot \frac{24}{5} = \underline{\frac{36}{25}}$

$E(Y_1 | Y_2 = 1) = \int_1^2 y_1 f(y_1 | y_2 = 1) = \int_1^2 y_1 \frac{2}{3} y_1\, dy_1 = \frac{2}{3}\int_1^2 y_1^2 dy_1 = \frac{2}{3} \cdot \frac{1}{3} y_1^3 \Big|_1^2 = \frac{2}{9}(8-1) = \underline{\frac{14}{9}}$

(g) No, Y_1, Y_2 are not independent, since $f(y_1,y_2) \ne f_1(y_1) f_2(y_2)$. Note also $E(Y_1 | Y_2 = 1) \ne E(Y_1)$

3 (a) $\underline{\text{Def}^n}$ $\text{Cov}(Y_1,Y_2) = E[(Y_1-\mu_1)(Y_2-\mu_2)]$, where $\mu_i = E(Y_i)$.

(b) Thus $\text{Cov}(Y_1,Y_1) = E[(Y_1-\mu_1)^2] = V(Y_1)$, the $\underline{\text{variance of } Y_1}$

(c) $E(U_1) = E(3Y_1 + 4Y_2) = 3E(Y_1) + 4E(Y_2) = 3(5) + 4(7) = 15+28 = \underline{43}$

(d) $V(U_1) = 3^2 V(Y_1) + 4^2 V(Y_2) + 2(3)(4)\,\text{Cov}(Y_1,Y_2) = 9(3) + 16(4) + 24(-2) = 27+64-48 = \underline{43}$

(e) $Cor(U_1, U_2) = Cov(3Y_1 + 4Y_2,\ 2Y_1 - Y_2)$

$$= (3)(2)\,Cov(Y_1, Y_1) + (3)(-1)\,Cov(Y_1, Y_2) + (4)(2)\,Cov(Y_2, Y_1) + (4)(-1)\,Cov(Y_2, Y_2)$$

$$= 6\,V(Y_1) + (-3+8)\,Cov(Y_1, Y_2) - 4\,V(Y_2) = 6(3) + 5(-2) - 4(4) = 18 - 10 - 16$$

$$= \underline{\underline{-8}}$$

4. Note $Y_1, Y_2 \sim N(0,1)$ (standard normal)

$$f(y_1, y_2) = \frac{1}{\sqrt{3}\,\pi} e^{-\frac{2}{3}(y_1^2 - y_1 y_2 + y_2^2)} \quad \& \quad f_2(y_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y_2^2}. \text{ Thus}$$

$$f(y_1|y_2) = \frac{f(y_1,y_2)}{f_2(y_2)} = \frac{\frac{1}{\sqrt{3}\,\pi} e^{-2/3(y_1^2 - y_1 y_2 + y_2^2)}}{\frac{1}{\sqrt{3\pi}} e^{-1/2\, y_2^2}} = \frac{1}{\sqrt{3}\pi \frac{1}{\sqrt{2}\sqrt{\pi}}} e^{-\frac{2}{3}y_1^2 + \frac{2}{3}y_1 y_2 - \frac{2}{3}y_2^2 + \frac{1}{2}y_2^2}$$

$$= \frac{1}{\sqrt{3\pi}\,\frac{1}{\sqrt{2}}} e^{-2/3\, y_1^2 + 2/3\, y_1 y_2 - 1/6\, y_2^2} = \frac{1}{\sqrt{2\pi}\,\frac{\sqrt{3}}{\sqrt{2}}\,\frac{1}{\sqrt{2}}} e^{-\frac{1}{2}\left(\frac{4}{3}y_1^2 - \frac{4}{3}y_1 y_2 + \frac{1}{3}y_2^2\right)}$$

$$= \frac{1}{\sqrt{2\pi}\,\frac{\sqrt{3}}{2}} e^{-\frac{1}{2}\left(\frac{y_1^2 - y_1 y_2 + 1/4\, y_2^2}{3/4}\right)} = \frac{1}{\sqrt{2\pi}\,\frac{\sqrt{3}}{2}} e^{-\frac{1}{2}\left(\frac{y_1 - \frac{1}{2}y_2}{\sqrt{3}/2}\right)^2},$$

σ μ

which is the density for a distribution which is normal with $\mu = \frac{1}{2}y_2$ & $\sigma = \frac{\sqrt{3}}{2}$.

5.(a) Y is proportion of hours lathe is in use $\Rightarrow$ $40Y$ is # hrs lathe is in use

$\therefore U = 40 - 40Y = 40(1-Y)$ is # hrs lathe is NOT in use.

(b) $F_U(u) = P(U \le u) = P(40(1-Y) \le u) = P\left(1 - Y \le \frac{u}{40}\right) = P\left(Y \ge 1 - \frac{u}{40}\right)$

Note

$$= P\left(Y \ge \frac{40-u}{40}\right) = \int_{\frac{40-u}{40}}^{1} 3y^2\,dy = y^3\Big|_{\frac{40-u}{40}}^{1} = \begin{cases} 0 & u \le 0 \\ 1 - \left(\frac{40-u}{40}\right)^3 & 0 < u < 40 \\ 1 & u \ge 40. \end{cases}$$

(c) $\therefore f_U(u) = \frac{d}{du} F_U(u) = -3\left(\frac{40-u}{40}\right)^2\left(-\frac{1}{40}\right)$

$$= \begin{cases} \frac{3}{40}\left(\frac{40-u}{40}\right)^2 & 0 < u < 40 \\ 0 & \text{o/wise}. \end{cases}$$

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