Transvection reduction: the link between determinants and the special linear group (expository)

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Abstract

It has long been known that row reduction is one of the most effective means for finding the determinant of a square matrix and that the special linear group is defined as the subgroup of the general linear group consisting of all elements with determinant one. Let's turn this around so that we define the determinant and the special linear group using transvections only. For notation let F be a field, let F^n be the vector space we study, and let $\mathrm{GL}(n, F)$ and $\mathrm{SL}(n, F)$ be the appropriate general linear group and special linear group. Also, let I_n be the $n \times n$ identity matrix.

Definition. An elementary row operation ρ is called a *transvection* if ρ replaces a row of a matrix by the sum of that row and a scalar multiple of another row. Each elementary row operation can be thought of as a linear transformation from F^n to F^n and has a matrix, $\rho(I_n)$, that carries out the elementary row operation. If ρ is a transvection, its corresponding matrix is also called a transvection (matrix).

One of the fundamental results about transvections is that for any $n \times n$ matrix A there is an $n \times n$ invertible matrix L that is a product of transvections such that $LA = U$ is an upper triangular matrix. The matrices L and U are not unique. There is a stronger result.

Theorem. *Let A be any $n \times n$ invertible matrix. There is an $n \times n$ invertible matrix L that is a product of transvections such that $LA = D$ is a diagonal matrix with ones in the first $n - 1$ entries and the last entry is a non-zero element $\delta \neq 0$ of F. If Δ is the set of all of these diagonal matrices in $\mathrm{GL}(n, F)$, then Δ is a subgroup of $\mathrm{GL}(n, F)$.*

Remember that we have not defined the determinant of a matrix yet, so how shall we define $\mathrm{SL}(n, F)$?

Definition. $\mathrm{SL}(n, F)$ is the subgroup of $\mathrm{GL}(n, F)$ generated by all transvection matrices.

Theorem. *$\mathrm{SL}(n, F)$ is a normal subgroup of $\mathrm{GL}(n, F)$ and $\mathrm{GL}(n, F) = \mathrm{SL}(n, F)\Delta$ is the join of $\mathrm{SL}(n, F)$ and Δ.*

Definition. Let A be an $n \times n$ matrix and take a product of transvections L so that $LA = U$ is an upper triangular matrix. The determinant of A, abbreviated $\det(A)$, is defined to be the product of the diagonal elements of U. If U happens to be in Δ with last entry δ, then $\det(A) = \delta$.

There is a problem with this definition of determinant similar to the uniqueness problem of the reduced row echelon form of a matrix.

Theorem. *$\mathrm{GL}(n, F)$ is a split extension of $\mathrm{SL}(n, F)$ by Δ if and only if the determinant function has $\det(A)$ determined uniquely.*

These group results do not appear directly in my linear algebra materials, but do appear indirectly and facilitate the derivation of the standard determinant results.

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