Two more variations on a theme of Desmond MacHale

Gabriela Mendoza

Abstract

In 1981 Desmond MacHale published an article entitled *Minimum counterexamples in group theory* in which he states 47 conjectures all known to be false, and asks for minimal counterexamples. In the tradition of MacHale we propose the following two conjectures which are obviously false.

Conjecture 1. *For a given prime p, the set of elements of order dividing p always forms a subgroup.*

Conjecture 2. *For a given prime p, the Sylow p-subgroup of a group is always normal.*

To make our notions more precise, we start with the following definition.

Definition 1. For a given prime p, a group is called *p-closed* if its Sylow p-subgroup is normal, and a group is called *minimal non-p-closed* if it is not p-closed but every subgroup and homomorphic image is.

For minimal non-p-closed groups we have the following result.

Theorem 2. *Let G be a minimal non-p-closed group. Then either G is simple or G has order pq^n, where p and q are distinct primes, and if Q is a Sylow q-subgroup of G, then Q is a minimal normal subgroup of G, where Q is an elementary abelian group of rank n, and if P is a Sylow p-subgroup of G, then P is cyclic of order p, $\mathrm{N}_G(P) = P$, and $q^n \equiv 1 \bmod p$.*

Denoting with $n(p)$ the order of the smallest minimal non-p-closed group, we obtain the following corollary to Theorem 2.

Corollary 3. *Let p be a prime, $p > 3$. Then $n(p) = \min\{p(kp+1), \frac{1}{2}p(p^2-1)\}$, where k is the smallest integer such that $kp+1$ is a prime power.*

We address now Conjecture 1 and start with the following definition.

Definition 4. For a given prime p, a group has property E_p if the elements of order dividing p form a subgroup, and a group is a *minimal non-E_p-group* if it is not an E_p group but every subgroup and homomorphic image is.

The following theorem relates minimal non-p-closed groups and minimal non-E_p-groups of smallest order.

Theorem 5. *Let p be a prime with $p > 3$. If G is a minimal non-p-closed group of smallest order, then G is a minimal non-E_p-group of smallest order. Every minimal non-E_p-group of smallest order is either a p-group or a minimal non-p-closed group of smallest order.*

Denoting with $f(p)$ the order of the smallest minimal non-E_p-group, we obtain the following corollary.

Theorem 6. *Let p be a prime with $p \geq 5$. Then $f(p) = \min\{p(kp+1), \frac{1}{2}p(p^2-1)\}$, where k is the smallest integer such that $kp+1$ is a prime power.*

Noting that for $p = 2, 3$ we get that $f(p) = n(p)$, we obtain the following surprising result.

Theorem 7. *For any prime p, $f(p) = n(p)$.*

This is joint work with Luise-Charlotte Kappe and Michael Ward.

Binghamton University, State University of New York
`mendoza@math.binghamton.edu`